Chapter 1

Lecture 1. Yang-Mills Theory

Prerequisites: (a) Canonic quantization in ϕ^4 theory and QED; (b) tree Feynman graphs; (c) basics of Lie groups (mostly $SU(N)$), generators, representations – fundamental and adjoint – group structure constants.

In the previous semester we covered basics and general aspects of quantum field theory. In this semester we will consider some particular, the most important field theories such as non-Abelian gauge theories (also known as Yang-Mills theories). During the last six decades, Yang-Mills theory has become the cornerstone of theoretical physics. It is seemingly the only fully consistent relativistic quantum field theory in four space-time dimensions. As such, it is the underlying theoretical framework for the Standard Model of Particle Physics (a part of which is the Glashow-Weinberg-Salam, GWS) model, which was proven to be the correct theory at all currently measurable energies. For recommended primary textbooks, see [1].

A few words are in order here as a warm-up introduction. Theoretical physics is an enormous subject, arguably, the most important fundamental science of nature. It is convenient to classify it using the so-called "magic $cG\hbar$ cube" invented and discussed in the late 1920s and early 1930s [2; 3]. It is shown in Fig. 1.1. Here c is the speed of light in vacuum. (Also, it is the maximal velocity of any object in nature.) It measures the extent of relativity. Next, G is the Newton constant. It normalizes gravity. Finally, analogously to c, the quantity $\hbar$ is another fundamental constant, the so-called Planck constant. It tells us when classical physics is overtaken by quantum physics. When you come to take this lecture course, you are supposed to already know all but two branches of theoretical physics indicated in Fig. 1.1.[1] The subject of my course is the phenomena which occur in systems with

[1] The back right upper corner is, perhaps, problematic.

typical velocities close to c and typical actions of the order of $\hbar$. This is the front right lower corner of the cube. Near the front right upper corner gravity effects become nonperturbative. This corner is supposed to be described by a future theory. Perhaps, it will be string theory, or something else, we do not know. And I will not venture into this territory.

In this course, as in its first part, I will use the system of units in which $c = \hbar = 1$. If so, energy and momentum have dimension of mass while length and spatial coordinates have dimension 1/mass. The Newton constant then defines a "fundamental" mass, also known as the Planck mass,

$$m_{\rm P} = \sqrt{\hbar c/G} \approx 1.22 \times 10^{19}\,\text{GeV}\,, \tag{1.1}$$

or, given that $c = \hbar = 1$,

$$G = \frac{1}{m_{\rm P}^2}\,. \tag{1.2}$$

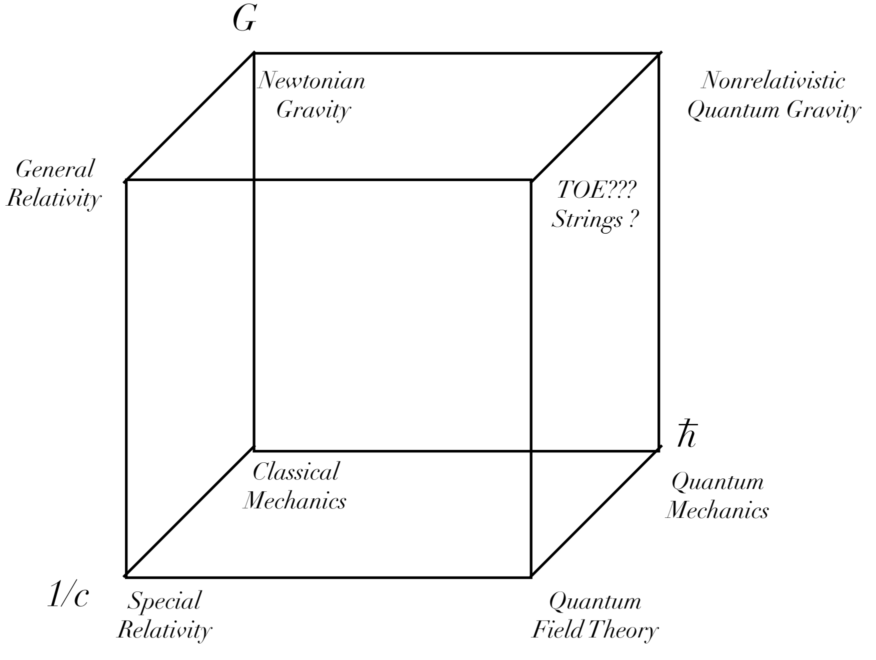

Fig. 1.1 The $cG\hbar$ cube of physics.

1.1 Construction of non-Abelian gauge theories

As you remember, gauge symmetry is *not* a symmetry but, rather, a redundancy in the description of the theory occurring when one elevates a global symmetry to the status of local symmetry. Let us review first the example of scalar QED (i.e. quantum electrodynamics of one scalar complex field ϕ).

Let us start with the globally U(1) invariant theory [2]

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - V(|\phi|)\,, \tag{1.3}$$

where V is the potential,

$$V = -m^2 \phi^\dagger \phi + \frac{\lambda}{2}(\phi^\dagger \phi)^2\,. \tag{1.4}$$

This theory has a symmetry under U(1) rotations,

$$\phi \to e^{i\alpha}\,\phi\,, \qquad \alpha = \text{const.} \tag{1.5}$$

Correspondingly, the theory has a continuous vacuum manifold. Any point

$$\phi_{\rm vac} = e^{i\alpha}\, v \equiv e^{i\alpha}\,\frac{m}{\sqrt{\lambda}} \tag{1.6}$$

is a valid vacuum (ground state). Above I assumed that the parameters m and λ are real and positive.

Now we would like to make the above theory invariant under local transformations with the phase $\alpha(\vec{x}, t)$. The potential V is obviously invariant. However, the kinetic term is not. To make it invariant we must add the photon field A_μ, and replace the partial derivative by a covariant derivative, $\partial^\mu \to D^\mu$, such that ϕ and $D^\mu\phi$ transform in one and the same way, namely,

$$\phi_{\rm gt}(x) = e^{i\alpha(x)}\,\phi(x)\,, \qquad \left(D^\mu\phi(x)\right)_{\rm gt} = e^{i\alpha(x)}\,\left(D^\mu\phi(x)\right)\,, \tag{1.7}$$

where the subscript gt stands for gauged transformed, and the four-coordinate x^μ for brevity is written as x with the superscript omitted, $x \leftrightarrow \{t,\, \vec{x}\}$. The second equality in (1.7) can be viewed as a basic definition of the covariant derivative. It is obviously satisfied if

$$D^\mu = \partial^\mu - iA^\mu \tag{1.8}$$

and the field $A^\mu(x)$ transforms as

$$A^\mu_{\rm gt} = A^\mu + \partial^\mu\alpha(x)\,. \tag{1.9}$$

[2]The metric used throughout this text is $g^{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$.

Combining (1.5), (1.8) and (1.9) it is very easy to check that (1.7) is valid. In the case at hand we certainly know (from the classical theory of electromagnetism) that the field A^μ which appeared in the process of "gauging"[3] is the electromagnetic four-potential. Its kinetic term is proportional to $F_{\mu\nu}F^{\mu\nu}$ where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{1.10}$$

Let us note the following identity relating $F_{\mu\nu}$ to a commutator[4] of covariant derivatives:

$$F^{\mu\nu} \equiv i\,[D^\mu D^\nu]\,. \tag{1.11}$$

It will help us carry out generalization to non-Abelian gauging.

Finally, the full Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{4e^2}\,F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi^\dagger)(D^\mu\phi) - V(|\phi|)\,, \tag{1.12}$$

where e is the coupling constant. From (1.8)–(1.10) it is clear that

$$\left(F^{\mu\nu}\right)_{\rm gt} = F^{\mu\nu}\,, \tag{1.13}$$

implying that the Lagrangian (1.12) is fully invariant under the local U(1) transformations presented in (1.7) and (1.8).

The phase of ϕ no longer presents a physical degree of freedom; rather, it is absorbed in photon's longitudinal polarization. (Remember, the photon acquires a mass provided the right-hand side in (1.6) does not vanish; this phenomenon is called Higgsing.) Simultaneously, the continuous vacuum manifold (1.6) shrinks into a single point since one can always choose, say, $\alpha = 0$ in (1.6) through a gauge condition on the fields. All points in (1.6) are gauge equivalent. There are no massless particles in the spectrum of the gauged theory with potential (1.4), in contradistinction with the ungauged theory (1.3).

In general, all filed configurations related to each other by gauge transformations represent one and the same point in the space of fields. This is why gauging a global symmetry introduces redundancy.

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Following the above pattern let us generalize the idea of gauging to non-Abelian symmetries. Assume that the field ϕ in (1.7) now carries an index i,

$$\phi \to \phi^i\,, \tag{1.14}$$

[3]This is how the transition from global to local symmetry is referred to.

[4]It is assumed that the derivatives act on arbitrary complex functions.

where for definiteness[5] we will choose i to be the index of the fundamental representation of SU(N),

$$i = 1, 2, ..., N\,. \tag{1.15}$$

Then

$$\phi^\dagger \to \phi_i^\dagger\,, \tag{1.16}$$

belongs to the antifundamental representation.

The Lagrangian (1.3) takes the form

$$\mathcal{L} = (\partial\phi_i^\dagger)(\partial\phi^i) - V(\phi_i^\dagger\phi^i)\,. \tag{1.17}$$

It is easy to see that it is invariant under a *global* SU(N) transformation. Indeed, if

$$\phi \to U\phi\,, \quad \phi^\dagger \to \phi^\dagger U^\dagger\,, \tag{1.18}$$

where U is any constant unitary matrix with unit determinant, $U \in$ SU(N), then

$$\begin{aligned} &\phi_i^\dagger\phi^i \to \phi^\dagger U^\dagger\, U\,\phi \to \phi^\dagger\phi\,; \\ &(\partial\phi_i^\dagger)(\partial\phi^i) \to (\partial\phi^\dagger)U^\dagger U(\partial\phi) \to (\partial\phi^\dagger)(\partial\phi)\,. \end{aligned} \tag{1.19}$$

The invariance of the Lagrangian (1.17) is obvious.

Now we want to make SU(N) local. Note that matrix $U \in$ SU(N) can be represented as

$$U = \exp\left(i\omega^a T^a\right), \tag{1.20}$$

where T^a are the generators of SU(N) and ω^a are arbitrary parameters which may or may not be x dependent, $a = 1, 2, ..., N^2 - 1$. For SU(2) the generators in the fundamental representation are proportional to the Pauli matrices, and for SU(3) proportional to the Gell-Mann matrices. In both cases the proportionality coefficient is 1/2. The standard normalization of the generators in the fundamental representation is

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab} \tag{1.21}$$

for any SU(N). The defining commutation relations for the generators (in any representation) are

$$[T^a T^b] = i f^{abc} T^c\,, \tag{1.22}$$

[5] The procedure is absolutely general and works for any representation of any non-Abelian group.

where f^{abc} are the group structure constants. For SU(2) one has $f^{abc} \equiv \varepsilon^{abc}$, where ε^{abc} is the Levi-Civita antisymmetric tensor.[6]

The global invariance implies $U =$ const by definition, which means, in turn that all ω^as in (1.20) are x independent. Now we want U to depend on the space-time point, $U \to U(x)$. Correspondingly,

$$\omega^a \to \omega^a(x)\,, \qquad a = 1, 2, ..., N^2 - 1\,. \tag{1.23}$$

The potential term in (1.17) remains invariant under the local SU(N) transformation. We want to generalize the kinetic term to be invariant under any local (x-dependent) transformation.

To this end we need to define the covariant derivative in such a way that, as in (1.7), after the gauge transformation (1.18),

$$\left(D^\mu \phi(x)\right)_{\rm gt} = U(x)\ \left(D^\mu \phi(x)\right)\,. \tag{1.24}$$

The solution to this equation is more contrived than in the Abelian theory because of non-commutativity of different matrices $U(x) \in \mathrm{SU}(N)$.

Let us assume (the assumption to be justified *a posteriori*) that

$$D_\mu = \partial_\mu - i A_\mu^a T^a\,, \tag{1.25}$$

where A_μ^a are non-Abelian gauge fields called gauge bosons (analogs of the electromagnetic four-potential), and T^as represent N^2-1 generators of SU(N). In QCD the A_μ^a fields are called gluons, for historical reasons.

Equation (1.24) is satisfied provided that

$$A_{\rm gt}^{\mu\,a}(x)\, T^a = \underbrace{U(x) \left(A^{\mu\,a}(x)\right) T^a\, U^\dagger(x)}_{\text{homogenious term}} - \underbrace{i \left(\partial^\mu U(x)\right) U^\dagger(x)}_{\text{inhomogenious}}\,. \tag{1.26}$$

Indeed,

$$\begin{aligned}
&\left(D^\mu \phi(x)\right)_{\rm gt} = \partial^\mu \left(U(x)\phi(x)\right) \\
&-i \left[U(x) \left(A^{\mu\,a}(x)\right) T^a\, U^\dagger(x) - i \left(\partial^\mu U(x)\right) U^\dagger(x)\right] \left(U(x)\phi(x)\right) \\
&= \left(\partial^\mu U(x)\right)\phi(x) + U(x)\partial^\mu \phi(x) - iU(x) \left(A^{\mu\,a}(x)\right) T^a \phi(x) \\
&- \left(\partial^\mu U(x)\right)\phi(x) \\
&= U(x)\left[\partial^\mu \phi(x) - i\left(A^{\mu\,a}\right) T^a \phi(x)\right] = U(x) D^\mu \phi(x)\,.
\end{aligned} \tag{1.27}$$

[6]The Levi-Civita tensor is also called permutation symbol or totally antisymmetric symbol. Tullio Levi-Civita (1873-1941) was a famous Italian mathematician.

For what follows it is useful to note that if $\omega^a \ll 1$, i.e. the gauge matrix $U(x)$ is close to unity (see (1.20)) then the change of $A^{\mu\,a}$ under the gauge transformation is

$$\delta A^{\mu\,a} = \partial^\mu \omega^a + f^{abc} A^{\mu\,b} \omega^c = D^\mu \omega^a \,. \tag{1.28}$$

In non-Abelian gauge theories the product $(A^{\mu\,a})\,T^a$ is often written as A^μ. In this shorthand the SU(N) index a is hidden. One must presume its presence from the context.

Examining Eq. (1.26) we note that, in contradistinction with the Abelian gauge transformation, $A^{\mu\,a}_{\rm gt}(x)\,T^a$ contains two terms: a non-homogeneous term (the second term on the right-hand side) analogous to what we had in the Abelian case, and an extra homogeneous term specific for non-Abelian gauge theories.

Finally, we have to establish the kinetic term for the gauge bosons. We will proceed analogously to Eq. (1.11). Assuming the covariant derivatives that act on an arbitrary column of N complex functions, we obtain

$$\begin{aligned} i\,[D^\mu D^\nu] &= \partial^\mu A^\nu - \partial^\nu A^\mu - i\,[A^\mu A^\nu] \stackrel{\rm def}{=} G^{\mu\nu} \\ &= \left[\partial^\mu A^{\nu\,a} - \partial^\nu A^{\mu\,a} + f^{bca} A^{\mu\,b} A^{\nu\,c}\right] T^a \end{aligned} \tag{1.29}$$

The combination in the square brackets is referred to as the gauge field strength tensor, in analogy with the electromagnetic field strength tensor,

$$G^{\mu\nu\,a} \equiv \partial^\mu A^{\nu\,a} - \partial^\nu A^{\mu\,a} + f^{bca} A^{\mu\,b} A^{\nu\,c} \,. \tag{1.30}$$

The nonlinear term on the right-hand side is absent in the Abelian case; it is due to noncommutativity of the group generator matrices.

What is the gauge transformation of the gluon field strength tensor? It is easy to derive it from Eq. (1.29) taking into account that

$$D^\mu_{\rm gt} = U D^\mu U^\dagger \,. \tag{1.31}$$

Then we find that

$$G^{\mu\nu}_{\rm gt} = U G^{\mu\nu} U^\dagger \,, \tag{1.32}$$

which implies in turn that ${\rm Tr}(G^{\mu\nu}\,G_{\mu\nu})$ is gauge invariant.

As a result, the kinetic term of the non-Abelian gauge field theory can be written as

$$\mathcal{L} = -\frac{1}{2g^2}\,{\rm Tr}\,(G^{\mu\nu}\,G_{\mu\nu}) = -\frac{1}{4g^2}\left(G^{\mu\nu\,a}\,G^a_{\mu\nu}\right), \tag{1.33}$$

where g is the gauge coupling constant. The full Lagrangian including the gauge and matter fields is

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{Tr}\left(G^{\mu\nu}\,G_{\mu\nu}\right) + (D^\mu\phi_i)^\dagger(D_\mu\phi^i) - V(\phi_i^\dagger\phi^i)\,. \tag{1.34}$$

Equation (1.34) includes all relevant operators. I hasten to add, however, that beyond perturbation theory we should add one extra term (which is P and T odd), namely the so-called θ term,

$$\mathcal{L}_\theta = \frac{\theta}{16\pi^2}\mathrm{Tr}\left(G^{\mu\nu}\,\tilde{G}_{\mu\nu}\right) = \frac{\theta}{32\pi^2}\left(G^{\mu\nu\,a}\,\tilde{G}^a_{\mu\nu}\right)\,, \tag{1.35}$$

where

$$\tilde{G}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\tilde{G}^{\alpha\beta\,a} \tag{1.36}$$

is the dual field strength tensor. This term introduces a new parameter θ also known as the vacuum angle. Why the θ term is important only beyond perturbation theory and where it comes from is a separate (*albeit* important) topic to which we will turn much later, see page 240.

1.2 Fermion (quark) matter

In the above construction for pedagogical purposes I used the scalar field in the fundamental representation. In theories relevant to nature, as a rule, the matter sector consists of fermions. The fermion part of the Lagrangian is basically the same as in QED, with the exception of definition of the covariant derivative (see (1.25)),

$$\mathcal{L}_{\rm ferm} = \sum_f \bar{\psi}_f\left(i\,\gamma^\mu D_\mu - m_f\right)\psi^f\,, \tag{1.37}$$

where f is the flavor index, ψ is the Dirac fermion and m_f is the mass of the quark of flavor f in the representation R of the gauge group G. Equation (1.37) is written for quantum chromodynamics; it is assumed that all fermion fields are in the fundamental representation of the gauge group SU(N) (in actuality, SU(3)), and so are the generators of the gauge group in the covariant derivatives acting on the fermions. Should we anticipate generalizations?

The answer is positive. First, in the Standard Model we deal with the Weyl rather than Dirac fermions. This will be discussed in due time.

Second, the matter fields need not be necessarily in the fundamental representation. Generally speaking, we can consider matter fields in any representation. The only change is that in Eq. (1.25) for the covariant derivative acting on the given matter field we must take the generator matrices T^a in the appropriate representation. For instance, in supersymmetric Yang-Mills theory the fermions to be considered are in the adjoint representation.

For real representations of the gauge group G (e.g. the adjoint representation) the fermion fields in (1.37) can be Majorana fields.

1.3 Yukawa couplings

By definition, the Yukawa coupling is a three-field coupling: two of the fields involved are fermionic and one is a boson spin-zero field. Yukawa couplings exist only in special cases when the matter sector of the Yang-Mills theory at hand contains such fields that one can built a Lorentz-scalar gauge invariant operator from three fields. The mass dimension of this term must be four (or D in the general D-dimensional space). For instance, let us consider the SU(N) gauge theory with the Dirac fermions in the fundamental representation and a complex scalar field Φ^a in the adjoint representation. Then one could add to (1.34), (1.37) the Yukawa term

$$\mathcal{L}_{\text{Yukawa}} = h\left(\bar{\psi}\, T^a \psi\right) \Phi^a\,, \tag{1.38}$$

where T^a's are the generators of the SU(N) group in the fundamental representation and h is a Yukawa coupling constant. Another popular option is $\bar{\psi}\psi\,\Phi$ where Φ is a gauge singlet field. I advise you to play with various representations of fermions and ϕ fields to build a variety of Yukawa terms. You may also refresh your memory of the Yukawa terms in the Glashow-Weinberg-Salam theory.

Appendix 1.1: C. N. Yang and Robert Mills

Yang and Mills had developed Yang-Mills theory (in 1954) in the context of an attempt to describe the strong interactions of vector mesons, such as ρ mesons. The SU(2) gauge theory they found did not work for this purpose, since (as we now know) what was needed was an SU(3) theory of colored quarks and gluons which came only 20 years later.

C. N. Yang (1922 -) and Robert Mills (1927 - 1999)
at Stony Brook in 1999.

Fig. 1.2 In 1957 the Nobel Prize in Physics was awarded jointly to Chen Ning Yang and Tsung-Dao (T.D.) Lee "for their penetrating investigation of the so-called parity laws which has led to important discoveries regarding the elementary particles." Robert Laurence Mills (1927-1999) was a physicist, specializing in quantum field theory, the theory of alloys, and many-body theory. While sharing an office at Brookhaven National Laboratory, in 1954, Yang and Mills proposed what is now called Yang-Mills field. Their original goal was to describe ρ mesons as gauge bosons (i.e. gauge the isotopic symmetry of strong interactions). Mills became Professor of Physics at the Ohio State University in 1956, and remained there until his retirement in 1995.

In 1953, Pauli was interested in a six-dimensional theory of Einstein's field equations of general relativity along the lines suggested by Kaluza and Klein in five dimensions. He compactified two extra dimensions into two-dimensional sphere, which inevitably led him to SU(2) Yang-Mills theory. However, non-Abelian gauge bosons remain massless, and at that time the only massless fields known to physicists were photons and gravitons, plus the neutrino postulated by Pauli in 1931, and not yet discovered in 1953. Somewhere I read that, when asked why he did not publish his research, Pauli said: "I have already introduced one hypothetical massless particle, and I had no nerve to introduce more..." (see [4], Chapter 1).

Because of his super-high requirements for his own work in physics, he put on hold publication on Yang-Mills, see below. In the meantime the theory was independently developed by C. N. Yang and Robert Mills, reported at a Princeton seminar and published in *Physical Review.*

Yang recollects of a seminar he gave in Princeton where Pauli was very critical and Pais was also present. Pauli was asking Yang about the mass of the intermediate vector mesons (now gluons), probably knowing that they were massless and therefore a killer for the theory (there are no massless hadrons). Yang responded he wasn't sure of the answer. Apparently, Pauli was so insistent and hostile with his questions that Yang just sat down at the front row and stopped talking! Then Oppenheimer encouraged him to continue delivering his talk, which he did.

Pauli described his SU(2) version of Yang-Mills theory, before Yang and Mills, in the letter to Abraham Pais [1] (page 171), entitled "Meson Nucleon Interaction and Differential Geometry" (written "to see what it looks like," in three days in July, (July 22-25 1953). See also N. Straumann, [6].

References

[1] L. D. Faddeev and A. A. Slavnov, *Gauge Fields: An Introduction To Quantum Theory,* Second Edition, (CRC Press, 1993);
Pierre Ramond, *Field Theory: A Modern Primer*, Second Edition, (Addison-Wesley, 1990).

[2] G. Gamov, D. Ivanenko and L. Landau, Zh. Russ. Fiz. Khim. Obstva. Chast Fiz. **60**, 13 (1928), (in Russian).

[3] M. Bronshtein, *K voprosu o vozmozhnoy teorii mira kak tselogo* [On a possible theory of the world as a whole], in *Osnovnye problemy kosmicheskoy fiziki* [*Basic problems of cosmic physics*], Kiev, ONTI (1934), pp. 186-218, and, in particular p. 210 (in Russian); M. Bronshtein, Physikalische Zeitschrift der Sowjetunion, **9**, 140 (1936).

[4] M. Shifman, *Standing Together in Troubled Times: Unpublished Letters by Pauli, Einstein, Franck and Others*, (World Scientific, 2017).

[5] O'Raifeartaigh, *Dawning of Gauge Theory*, (Princeton University Press, 1997).

[6] N. Straumann, *On Pauli's Invention of non-Abelian Kaluza-Klein Theory in 1953*, http://arxiv.org/pdf/gr-qc/0012054.pdf