Preface

This book is a sequel to the book [1] by the same authors entitled *Theory of Groups and Symmetries: Finite Groups, Lie Groups, and Lie Algebras*, so this book can be viewed as the second part. References to sections and formulas from the first book are labeled by Roman numeral I, e.g., Section I-3.2.3, Eq. (I-7.2.37).

The second book is mainly devoted to the theory of representations of Lie groups and Lie algebras and also to applications. We begin with the presentation in Chapter 1 of the Dirac notation, which is illustrated by boson and fermion oscillator algebras and also Grassmann algebra. In Chapter 2, we give a detailed account of finite-dimensional representations of groups $SL(2,\mathbb{C})$ and $SU(2)$ and their Lie algebras. In Chapter 3, we consider the general theory of finite-dimensional irreducible representations of simple Lie algebras based on the construction of highest weight representations. We then give the classification of all finite-dimensional irreducible representations of the Lie algebras of the classical series $s\ell(n,\mathbb{C})$, $so(n,\mathbb{C})$ and $sp(2r,\mathbb{C})$.

In Chapter 4, finite-dimensional irreducible representations of linear groups $SL(N,\mathbb{C})$ and their compact real forms $SU(N)$ are constructed on the basis of the Schur–Weyl duality. In this approach, spaces of irreducible representations are viewed as invariant subspaces in the tensor products $\mathcal{V}^{\otimes r}$ of spaces $\mathcal{V}$ of defining representations. These invariant subspaces are singled out with the help of special elements (idempotents) of group algebra $\mathbb{C}[S_r]$ of the group of permutations (symmetric group) S_r. Here, special role is played by the theory of representations of the algebra $\mathbb{C}[S_r]$ (Schur–Frobenius theory, Okounkov–Vershik approach), based on combinatorics of Young diagrams and Young tableaux.

We then turn in Chapter 5 to pseudo-orthogonal groups $O(p,q)$ and $SO(p,q)$, including multidimensional Lorentz groups $O(1,N-1)$ and $SO(1,N-1)$ and their Lie algebras. We also consider symplectic groups $Sp(p,q)$, albeit in less detail. The presentation here is based on the appropriate modification of the techniques exposed in Chapter 4.

Finally, in Chapter 6 we study the covering groups $\mathsf{Spin}(p,q)$ for pseudo-orthogonal groups $SO^{\uparrow}(p,q)$. The groups $\mathsf{Spin}(p,q)$ are called spinor groups and are currently in active use in quantum field theory. The construction of the spinor groups $\mathsf{Spin}(p,q)$ requires the introduction of Clifford algebras in spaces $\mathbb{R}^{p,q}$ and the discussion of the representations of these algebras.

In this book, as done in the first one, we try to prove or at least give hints of proofs for the majority of facts. We emphasize, however, that, similar to the first book, the level of rigor in places is not to the standard of a mathematically oriented reader; our main goal is rather to make the material comprehensible. Text written in small font may be omitted at the first reading. When necessary, we give references to literature, where one can find more detailed analysis of one or another point.

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