On Affine Fibrations

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Abstract

In this survey article we highlight some recent developments in the problems on affine fibrations. We show how some recent results completely determine the structure of an $\mathbf{A}^1$ fibration over a seminormal ring and how they can be used to prove a generalised epimorphism theorem. We also show how the problem of $\mathbf{A}^2$ fibration over a two-dimensional regular local ring is related to the problem of embedding of a plane in affine 3-space over a discrete valuation ring.

1 Introduction

For a commutative local ring S and an S-algebra A, it is useful to know in affine algebraic geometry, sufficient criteria for deciding whether A is a polynomial ring in n variables over S. The following conditions are obviously necessary :

(i) A is a finitely generated flat S-algebra.

(ii) For every prime ideal P of S, A_P/PA_P is a polynomial ring in n variables over S_P/PS_P.

(iii) $\Omega_{A/S}$ is a free A-module of rank n.

One would like to know when these conditions are sufficient. In this article we make a brief survey of recent developments in this direction.

We first recall some standard notations to be used throughout the article. For a commutative ring S, $S^{[n]}$ denotes a polynomial ring in n variables over S. For a prime ideal P of S, $k(P)$ denotes the field S_P/PS_P. For an S-algebra A, $\Omega_{A/S}$ is the module of Kahler differentials of A over S.

2 General affine fibrations

Definition 2.1. For a commutative ring S, an S-algebra A is said to be an affine n-fibration over S (denoted by $\mathbf{A}^n$), if the following conditions hold:

(i) A is a finitely generated flat S-algebra.

(ii) For every prime ideal P of S, $A \otimes_S k(P) = k(P)^{[n]}$.

Example 2.2. For a ring S, let $B = S^{[\ell]}$, M a projective module of rank m over B and let $A = Sym_B(M)$. Let $n = \ell + m$. By the Quillen - Suslin theorem ([13, Theorem 3] and [18, Theorem 1]) any projective module over $k(P)^{[\ell]}$ is free and hence it is easy to see that A is an $\mathbf{A}^n$-fibration over S.

The following remarkable theorem of T.Asanuma gives a complete structure theorem for an $\mathbf{A}^n$-fibration over a noetherian ring S ([2, Theorem 3.4).

Theorem 2.3. *Let S be a noetherian ring and A an $\mathbf{A}^n$-fibration over S. Then $\Omega_{A/S}$ is a projective A-module of rank n and A is (upto an isomorphism) an S-subalgebra of a polynomial ring $S^{[m]}$ for some m such that*

$$A^{[m]} \cong Sym_{S^{[m]}}(\Omega_{A/S} \otimes_A S^{[m]}).$$

If moreover $\Omega_{A/S}$ is a free A-module (of rank n) then $A^{[m]} = S^{[m+n]}$.

The general question of when an $\mathbf{A}^n$-fibration is an affine $\mathbf{A}^n$-bundle (i.e., locally a polynomial ring in n variables) was first raised by I.V. Dolgachev and B.Weisfeiler ([9]). As a consequence of Asanuma's theorem it follows ([2], 3.5.) :

Corollary 2.4. *Let S be a regular local ring. Then an $\mathbf{A}^n$ fibration A over S is at least a stably polynomial algebra over S, i.e., $A^{[t]} = S^{[n+t]}$ for some integer $t \geq 0$.*

However, if S is not a regular local ring, then an $\mathbf{A}^n$-fibration over S need not be a stably polynomial algebra over S as the following examples illustrate :

Example 2.5. ([20, 4.1]). Let k be a field of characteristic zero. Let $S = k[[t^2, t^3]]$ and $A = S[X + tX^2] + (t^2, t^3)S[X]$. Then one can check that A is an $\mathbf{A}^1$-fibration over S but A is not a stably polynomial algebra over S.

Example 2.6. Let k be a field, $S = k[[t^2, t^3]], I = (1 - t^2X^2, t^2 - t^3X)S[X]$. One can check that I is a projective $S[X]$-module of rank 1 which is not free. Let $A = Sym_{S[X]}(I)$. Then A is an $\mathbf{A}^2$-fibration over S. But A is not a stably polynomial algebra over S.

Example 2.7. Let $S = \mathbf{C}[X,Y,Z]_{(X,Y,Z)}/(X^4 + Y^4 + Z^4)$. Then S is a normal local domain of dim 2. V. Srinivas has shown ([17, Sect. 3]) that there exists a projective module P of rank 2 over $S[X]$ which is not stably free. Let $A = Sym_{S[X]}(P)$. Then A is an $\mathbf{A}^3$-fibration over S which is not a stably polynomial algebra over S.

3 $\mathbf{A}^1$-fibration

We now discuss in detail the problem of $\mathbf{A}^1$-fibration. We first mention two results of E. Hamann ([10, Theorems 2.6 and 2.8]) and R.G.Swan ([19, Theorem 6.1]) respectively.

Theorem 3.1. *If S_{red} is a noetherian seminormal ring or if S contains the field of rationals, then $S^{[1]}$ is S-invariant (i.e., if $S \subseteq A$ and $A^{[m]} = S^{[m+1]}$, then $A = S^{[1]}$).*

Theorem 3.2. *If S_{red} is a seminormal ring, then Pic $(S^{[m]})$ = Pic (S) for all $m \geq 1$.*

From the theorems (2.3), (3.1) and (3.2) it is easy to deduce

Theorem 3.3. *Let S be a noetherian local ring such that S_{red} is seminormal. Let A be an $\mathbf{A}^1$- fibration over S. Then $A = S^{[1]}$.*

Thus using a result of Bass-Connell-Wright (see (4.7)) one concludes that *if A is an A^1-fibration over a noetherian seminormal ring S, then $A \cong Sym_S(P)$ for a projective S-module P of rank 1.*

On the other hand, if S_{red} is not seminormal then J.Yanik's example (2.5) shows than an $\mathbf{A}^1$-fibration A over S need not be even a stably polynomial algebra over S. However if $\Omega_{A/S}$ is free then by (2.3), $A^{[m]} = S^{[m+1]}$. Therefore using (3.1) we have

Theorem 3.4. *Let S be a noetherian ring such that either S contains the field of rationals or S_{red} is seminormal. Let A be an $\mathbf{A}^1$-fibration over S such that $\Omega_{A/S}$ is a free A-module. Then $A = S^{[1]}$.*

The necessity of the condition on $\Omega_{A/S}$ has been illustrated by example (2.5). E.Hamann has given an example in ([10]) to illustrate the necessity of the condition on S. We give below a similar example.

Example 3.5. Let $R = \mathbf{Z}_{(2)}[2\sqrt{2}]$, $A = R[X,Y]$, $S = R[F]$ where

$$F = X - 2Y(\sqrt{2}X - Y^2) + \sqrt{2}(\sqrt{2}X - Y^2)^2 - \sqrt{2}(Y - \sqrt{2}(\sqrt{2}X - Y^2))^4.$$

It has been shown in ([6], 4.1) that A is an $\mathbf{A}^1$-fibration over S and $\Omega_{A/S}$ is free but $A \neq S^{[1]}$ (in fact $R[X,Y]/(F) \neq R^{[1]}$).

In the context of the above example we remark that in ([6], Theorem 3.1) it was shown that *for any noetherian ring R and an element $F \in R[X,Y]$ which is algebraically independent over R, $R[X,Y]$ is an $\mathbf{A}^1$-fibration over $R[F]$ if and only if $R[X,Y]$ is a stably polynomial algebra over $R[F]$*. Also in view of Example (3.5) one may ask the following question :

Question 3.6. Let R be a noetherian domain. Let $F \in R[X,Y]$ be such that $R[X,Y]$ is an $\mathbf{A}^1$-fibration over $R[F]$ and $R[X,Y]/(F) = R^{[1]}$. Then is $R[X,Y] = R[F]^{[1]}$?

In connection with Question (3.6) we mention the following result of the first author ([5]).

Proposition 3.7. *For any commutative domain R of characteristic zero, if $F \in R[X,Y]$ is such that $R[X,Y]/(F) = R^{[1]}$, then $R[X,Y]$ is an $\mathbf{A}^1$-fibration over $R[F]$ and $\Omega_{R[X,Y]/R[F]}$ is a free $R[X,Y]$-module.*

Applying the results (2.3), (3.1) and (3.7) the following generalisation of the famous Abhyankar-Moh epimorphism theorem ([1,Theorem 1.2]) was obtained in ([5,Theorems 3.7 and 3.9]).

Theorem 3.8. *For a commutative domain of characteristic zero which is either seminormal or contains a field, $R[X,Y]/(F) = R^{[1]}$ implies that $R[X,Y] = R[F]^{[1]}$.*

In view of Asanuma's result that any $\mathbf{A}^1$-fibration over a noetherian ring S is contained in a polynomial ring $S^{[m]}$ for some m, one would like to know under what conditions an S-subalgebra of $S^{[m]}$ is an $\mathbf{A}^1$-fibration over S.

Recently the authors investigated the problem and the following results were obtained in ([8]).

Theorem 3.9. *Let S be a noetherian ring and A an S-subalgebra of $S^{[m]}$ such that*

(i) A is S-flat.

(ii) $A \otimes_S k(P)$ are factorial domains for all prime ideals P of S and dim $(A \otimes_S k(P)) = 1$ for all minimal prime ideals P of S.

Then A is an $\mathbf{A}^1$-fibration over S.

Theorem 3.10. *Let S be a noetherian ring containing a field of characteristic zero and A an S-subalgebra of $S^{[m]}$ such that*

(i) A is S-flat.

(ii) $A \otimes_S k(P)$ are 1-dimensional normal domains for all minimal prime ideals P of S.

(iii) $A \otimes_S k(P)$ are integral domains for all prime ideals P of height 1 in S.

Then A is an $\mathbf{A}^1$-fibration over S.

Consequently, under the hypotheses of either (3.9) or (3.10), and the further assumption that S_{red} is seminormal, using the results (2.3), (3.1), (3.2) and a theorem of Bass - Connell - Wright (see (4.7) below), one concludes that $A \cong Sym_S(M)$ where M is a projective S-module of rank 1. Examples have also been given in ([8]) to show that the conditions in (3.9) and (3.10) are necessary.

4 $\mathbf{A}^2$-fibration

We now discuss problems in $\mathbf{A}^2$-fibration. In general an $\mathbf{A}^2$-fibration over a noetherian ring S need not be even a stably polynomial algebra over S as Example (2.6) showed. However, if S is a regular local ring then by Asanuma's result (2.4), if A is an $\mathbf{A}^2$-fibration over S then at least $A^{[m]} = S^{[m+2]}$ for some integer $m \geq 0$. One would like to know if $A = S^{[2]}$.

In this direction, A.Sathaye first proved the following beautiful theorem over a discrete valuation ring S. ([16, Theorem 1]) :

Theorem 4.1. *Let S be a discrete valuation ring containing a field of characteristic zero. Let A be an $\mathbf{A}^2$-fibration over S. Then $A = S^{[2]}$.*

(Asanuma has shown in ([2, Theorem 3.1]) that it is not necessary to assume that A is finitely generated over S).

Asanuma has also given the following example in ([2, 5.1]) to show that it is necessary to assume in (4.1) that S contains a field of characteristic zero.

Example 4.2. Let $S = \mathbf{Z}_{(p)}$ for some prime integer p. Let

$$A = \mathbf{Z}_{(p)}[X,Y,Z]/(pZ - Y - Y^{pq} + X^{p^2})$$

where q is an integer ≥ 2 and $(p,q) = 1$. Then A is an $\mathbf{A}^2$-fibration over S. But $A \neq \mathbf{Z}_{(p)}^{[2]}$.

One would now like to know whether an $\mathbf{A}^2$-fibration over a regular local ring S of dimension ≥ 2 is necessarily $S^{[2]}$. In view of Example (4.2) one of course has to assume that S contains the field of rationals. The problem is still open. As a starting point we ask the following question :

Question 4.3. Let k be a field of characteristic zero. Let S be a regular two-dimensional affine k-spot (i.e., localisation of a regular affine k-algebra at a prime ideal). Let A be an $\mathbf{A}^2$-fibration over S. Then is $A = S^{[2]}$?

In ([12]) M.P. Murthy has shown that *if S is a regular local ring of dim 2 , then any projective module over $S^{[m]}$ is free* . Thus it follows from Asanuma's result (2.3) that in the situation of (4.3), $\Omega_{A/S}$ is a free A- module of rank 2.

We now show how Question (4.3) is connected with the following epimorphism problem over a discrete valuation ring :

Question 4.4. Let R be a discrete valuation ring containing a field of characteristic zero. Let $F \in R[X,Y,Z]$ be such that $R[X,Y,Z]/(F) = R^{[2]}$. Then is $R[X,Y,Z] = R[F]^{[2]}$?

We first state certain well-known results. The following result is due to

H.Lindel ([11]) :

Theorem 4.5. *Let k be a perfect field. Let S be a regular k-spot of dimension n. Then there exists a field L such that $k \hookrightarrow L \hookrightarrow S$ and an affine L-algebra B with maximal ideal M such that $S = B_M$ and B/MB is a finite separable extension of L. Moreover there exists a subring S_1 of S of the form $S_1 = L[X_1, \cdots, X_n]_{(X_1, \cdots, X_{n-1}, \phi(X_n))}$ and an element $t \in S_1$ such that the canonical map $S_1/tS_1 \to S/tS$ is an isomorphism (where ϕ is the irreducible polynomial of the simple field extension B/MB of L).*

The following result of the first author can be proved using ideas in ([4]).

Lemma 4.6. *Let $R \hookrightarrow S$ be noetherian domains such that R and S are analytically isomorphic along $t \in R$ (i.e., the canonical map $R/tR \to S/tS$ is an isomorphism). Given a finitely generated flat S-algebra A such that $A_t = S_t^{[n]}$, there exists a finitely generated flat R-algebra D such that $D_t = R_t^{[n]}$, $A \cong D \otimes_R S$ and $A/tA \cong D/tD$. If A is an $\mathbf{A}^n$-fibration over S, then D is an $\mathbf{A}^n$-fibration over R.*

Next we state a patching up theorem of Bass - Connell - Wright ([3, Theorem 4.4]) :

Theorem 4.7. *Let A be a finitely presented S-algebra. Suppose that for all maximal ideals M of S, the S_M-algebra A_M is isomorphic to the symmetric algebra of some S_M-module. Then $A \cong Sym_S(P)$ for a finitely presented S-module P.*

As a consequence of (4.1) and (4.7) and the fact that any projective module over a P.I.D is free, it is easy to see

Corollary 4.8. *Let S be a P.I.D. containing a field of characteristic zero. Let A be an $\mathbf{A}^2$-fibration over S. Then $A = S^{[2]}$.*

We now resume our discussion of Question (4.3). Recall that k is a field of characteristic zero, S a regular affine k-spot of dimension 2 and A an $\mathbf{A}^2$-fibration over S. Then by (4.5) there exists a field L in S containing k, an affine L-algebra B and a maximal ideal M of B such that $S = B_M$ and B/MB is a finite separable extension of L. Also there exists a subring S_1 of S and an element $t \in S_1$ such that

(i)$S_1 = L[U, V]_{(U, \phi(V))}$ where $B/MB = L[T]/(\phi(T))$.

(ii) $S_1/tS_1 \xrightarrow{\sim} S/tS$.

Now it is easy to see that S_t is a P.I.D. and A_t an $\mathbf{A}^2$-fibration over S_t. Thus by (4.8), $A_t = S_t^{[2]}$. Using (i) and (ii) above and applying (4.6), there exists a finitely generated flat algebra A_1 over S_1 such that $A_1[1/t] = S_1[1/t]^{[2]}$, $A \cong A_1 \otimes_{S_1} S$, $A_1/tA_1 \cong A/tA$ and A_1 is an $\mathbf{A}^2$-fibration over S_1.

Let $R = L[V]_{(\phi(V))}$. This is a discrete valuation ring with parameter π, say. Now $R[U] \hookrightarrow S_1$ and $R[U]/(U) \xrightarrow{\sim} S_1/(U)S_1$. Now $S_1[1/U]$ is clearly a P.I.D. and A_1 being an $\mathbf{A}^2$-fibration over S_1, $A_1[1/U]$ is also an $\mathbf{A}^2$-fibration over $S_1[1/U]$. Again applying (4.8), $A_1[1/U] = S_1[1/U]^{[2]}$. Therefore, by (4.6), there exists a finitely generated flat $R[U]$-algebra D such that

(a) D is an $\mathbf{A}^2$-fibration over $R[U]$.

(b) $D[1/U] = R[U, 1/U]^{[2]}$.

(c) $A_1 \cong D \otimes_{R[U]} S_1$ and hence $A \cong D \otimes_{R[U]} S$.

By (c) it follows that Question (4.3) would have an affirmative answer if we can show that $D = R[U]^{[2]}$. Unfortunately in the above situation (i.e., if D is an $R[U]$-algebra over a d.v.r. R satisfying (a) and (b)) we even do not know whether $D = R^{[3]}$ let alone being $R[U]^{[2]}$. So as a starting point we make the assumption that $D = R^{[3]}$. Note that the condition (a) is equivalent to

(a') $D_\pi = R_\pi[U]^{[2]}$ and $D/\pi D = (R/\pi R)[U]^{[2]}$.

We are thereby asking the following question :

Question 4.9. Let R be a discrete valuation ring containing the field of rationals. Let π be a parameter of R, $K = R_\pi$ and $k = R/\pi R$. Let $F \in R[X, Y, Z]$ be such that

(i) $K[X, Y, Z] = K[F]^{[2]}$.

(ii) $k[X, Y, Z] = k[\overline{F}]^{[2]}$.

(iii) $R[X, Y, Z, 1/F] = R[F, 1/F]^{[2]}$.

Then is $R[X, Y, Z] = R[F]^{[2]}$? (For a concrete example, see (4.13) below).

Note that by (4.1), the conditions (i) and (ii) together imply that $R[X,Y,Z]/(F) = R^{[2]}$. We are thus naturally led to the Question (4.4) raised earlier. Thus the $\mathbf{A}^2$-fibration problem over a two-dimensional regular k-spot involves a question of $\mathbf{A}^2$-fibration over $R[U]$ for a d.v.r. R which in turn involves an epimorphism question over a d.v.r.. The Question (4.4) is still open in general. The authors investigated the case of linear planes over a d.v.r. and obtained the following result ([7, 3.5]):

Theorem 4.10. *Let R be a discrete valuation ring (of any characteristic) with parameter π. Let $F \in R[X,Y,Z]$ be such that*

(i) $R[X,Y,Z]/(F) = R^{[2]}$.

(ii) $F = f(X,Y)Z - g(X,Y)$ *where* $f, g \in R^{[2]}$ *and* $f(X,Y) \notin \pi R[X,Y]$.

Then $R[X,Y,Z] = R[F]^{[2]}$.

This result may also be viewed as a generalisation of A. Sathaye's theorem on linear planes over a field ([15]) stated below.

Theorem 4.11. *Let K be a field (of any characteristic) and $F \in K[X,Y,Z]$ be such that*

(i) $K[X,Y,Z]/(F) = K^{[2]}$.

(ii) $F = f(X,Y)Z - g(X,Y)$ *where* $f, g \in K^{[2]}$ *and* $f \neq 0$.

Then $K[X,Y,Z] = K[F]^{[2]}$.

Note that in the situation of (4.4), if K is the quotient field of R, then we have $K[X,Y,Z]/(F) = K^{[2]}$. Thus to investigate (4.4) one would first ask if, for a field K of ch. 0, an element $F \in K[X,Y,Z]$ satisfying $K[X,Y,Z]/(F) = K^{[2]}$ is necessarily a variable in $K[X,Y,Z]$. The most general answer known so far is the following theorem of P. Russell and A. Sathaye ([14, 3.8.2]) :

Theorem 4.12. *Let K be a field of characteristic zero and let $F \in K[X,Y,Z]$ be such that*

(i) $K[X,Y,Z]/(F) = K^{[2]}$.

(ii) $F = \sum_{0 \le i \le n} f_i(X,Y)Z^i$ *with* $f_i \in K^{[2]}$ *where g.c.d.* $(f_1, \cdots, f_n)$ *is a non-unit in* $K[X,Y]$.

Then $K[X,Y,Z] = K[F]^{[2]}$.

We conclude our discussion by giving a concrete non-trivial example of a linear plane over a d.v.r..

Example 4.13. Let (R, π) be a discrete valuation ring containing the field of rationals. Let $K = R_\pi$ and $k = R/\pi R$. Define F, G, $H \in R[X,Y,Z]$ as

$$F = (\pi Y^2)Z + Y + \pi Y(X + X^2) + \pi^2 X.$$

$$G = \pi X + Y(YZ + X + X^2).$$

$$H = \pi^2 Z - (YZ + X + X^2)(\pi(2X+1) + Y(YZ + X + X^2)).$$

One can check that

(i) $K[X,Y,Z] = K[F,G,H]$ $(= K[F]^{[2]})$.

(ii) $k[X,Y,Z] = k[\overline{F}]^{[2]}$ (since $\overline{F} = Y$).

(iii) $R[X,Y,Z,1/F] = R[F,1/F,G]^{[1]} = R[F,1/F]^{[2]}$.

Note that by (i) and (ii) and Theorem (4.1), $R[X,Y,Z]/(F) = R^{[2]}$.

We end with the question

Question 4.14. In the above example, is $R[X,Y,Z] = R[F]^{[2]}$?

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